Finite-Difference Analysis of a Loaded Hemispherical Resonator

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***Abstract*—A rotationally symmetric inhomogeneously loaded open hemispherical resonator is analyzed using the finite-difference (FD) frequency-domain method. State-of-the art or new techniques are proposed to achieve high accuracy and efficiency of computations. These include applying the Galerkin method followed by an inexact shift-and-inverse Lanczos technique with an approximate starting eigenvector for selective computation of a single desired high-order mode, and compensating numerical dispersion error with the modified FD formulas. As a result, the final value of the computed resonant frequency for a desired mode agrees within 0.1% with the measured value and the computations involving a few 100 000 unknowns are carried out in minutes on a personal computer.**

***Index Terms*—Finite difference frequency domain (FDFD), hemispherical resonator.**

I. Motivation

INTENSIVE research in material science generates a growing need for efficient low-cost characterization and design of new media. One of widely applied characterization techniques is based on investigating the resonances excited (in the millimeter-wave frequency range) in open hemispherical resonators with insertions being the investigated media [1]–[3]. The high costs associated with expensive fabrication of advanced materials and repetitive measurements of different sample designs can be reduced if one were able to accurately and efficiently model or analyze such resonators. However, the existing analysis of hemispherical resonators, based on paraxial wave propagation approximation [4], does not allow one to study effects such as finite sample size, diffraction on mirror edges, nonideal mirror shape or coupling aperture. Moreover, the formulas relating the resonant frequencies with media parameters are valid only for a thin single-layer material samples. Clearly, this severely limits application of this simplified approach in practical situations.

Consequently, a proper description of field effects taking place in a hemispherical resonator with inhomogeneous insertions requires full-wave numerical modeling. Due to the large size of the cavity, this is an extremely difficult computational task if high accuracy is to be maintained. To the best of our knowledge, the only attempts thus far were based on the three-dimensional (3-D) boundary-element method (BEM) [5] and the staircased 3-D finite difference time domain (FDTD) [3]. Both methods are computationally intensive and the staircasing used in the FDTD introduces unacceptable errors. In the following sections, we present a low-cost approach toward modeling the discussed system, which effectively deals with the problems of the previously proposed algorithms.

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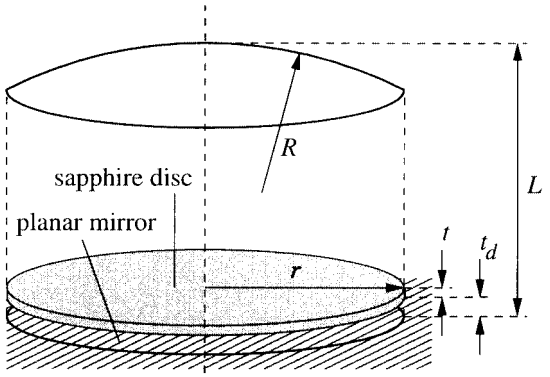

Fig. 1. Hemispherical resonator loaded with a thin sapphire disc. L is the length of the resonator, R is the radius of curvature of the spherical mirror, r is the radius of the sapphire disc, t is the thickness of the disc, and t_d is the distance between the disc and planar mirror.

II. High-Accuracy Numerical Modeling

A typical resonator (cf. Fig. 1) is $L = n_0 \cdot \lambda$ long, where $n_0 > 50$, and has at least $2r = 10 \cdot \lambda$ in diameter, where λ is the length of the wave in vacuum. For a structure of this size, a large matrix eigenvalue problem has to be solved in order to find field modes and corresponding resonant frequencies, and numerical dispersion may introduce unacceptable errors into computer simulations (e.g., [6]).

Moreover, the mode of interest is relatively high (e.g., quasi-$\mathrm{TEM}_{0,0,50}$ in the Gaussian beam notation), which means that one has to find eigenvalues located far from either of the spectrum ends. To overcome these difficulties, we applied the strategy summarized in the following points.

- First, the resonator was analyzed using an approximate method based on paraxial approximation and the Galerkin procedure (cf. Section III) in order to find a starting vector for a full-wave numerical solver.
- Using the rotational symmetry, the 3-D frequency-domain problem was expressed in cylindrical coordinates and reduced to two dimensions with three field components. Curved mirrors were modeled by means of conformal

technique of [7] and an effective permittivity concept was applied to dielectric boundaries [8]. The constructed wave equation was discretized using Yee's mesh and modified formulas accounting for numerical dispersion. As a result, a standard symmetrical eigenvalue problem was obtained with ω^2 being an eigenvalue (where ω is an unknown angular frequency) [9].

- An inexact shift and invert Lanczos (ISIL) procedure [10] with a previously computed starting vector was used to solve the full-wave eigenproblem.

In the following sections, we explain our strategy in more detail.

III. Approximate Analysis of a Loaded Resonator

A starting point in our analysis are modes of an empty resonator. For this configuration, the solution can be found in analytical form. Let us consider the wave equation for the electric field

$$\nabla^2 E(x, y, z) + k^2 E(x, y, z) = 0. \tag{1}$$

Assuming

$$E(x, y, z) = u(x, y, z)\, e^{-jkz} \tag{2}$$

(1) becomes

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - 2jk\frac{\partial u}{\partial z} = 0. \tag{3}$$

For a slowly varying envelope (paraxial approximation), i.e., if

$$\left|\frac{\partial^2 u}{\partial z^2}\right| \ll \left|2k\frac{\partial u}{\partial z}\right| \tag{4}$$

the wave equation reduces to

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - 2jk\frac{\partial u}{\partial z} = 0. \tag{5}$$

The solutions to the above equation can be found in an analytical form [11]. One series of such solutions are TEM_{00q} modes, which correspond to resonances of a Gaussian beam. The resonant frequencies of these modes can be found by considering the general solution of the paraxial wave equation in the cylindrical coordinate system. Assuming no variation in the angular direction, $u(\rho, z)$ is expressed by

$$u(\rho, z) = A_0 \frac{W_0}{W(z)} \exp\left(-\frac{\rho^2}{W^2(z)}\right) \cdot \exp\left(-jkz - jk\frac{\rho^2}{2R(z)} + j\zeta(z)\right) \tag{6}$$

where $W(z) = W_0\left[1 + (z/z_0)^2\right]^{1/2}$ is the beam radius, $W_0 = (\lambda z_0/\pi)^{1/2}$, $R(z) = z\left[1 + (z_0/z)^2\right]$ is the constant phase radius, $\zeta(z) = \arctan(z/z_0)$ is the phase difference between the plane wave and the Gaussian beam, $z_0 = \sqrt{R(R-L)}$ is the Rayleigh length, A_0 is the amplitude of the field, and finally, λ is the wavelength. The resonant frequencies for TEM_{00q} modes in the hemispherical resonator having length L and the radius of curvature of the upper mirror R are then given by

$$f_{00q} = \frac{c}{2L}\left[q + \frac{1}{\pi}\arccos\left(\sqrt{1 - \frac{L}{R}}\right)\right] \tag{7}$$

with c being the velocity of light in vacuum.

In order to find an approximate resonant frequency and modal field distribution for a resonator loaded with an inhomogeneous sample, we use the Galerkin method with the modes of an empty resonator used both as a set of basis and testing functions. Let us assume that the sample is homogeneous in both the x- and y-directions and the z-component of the $\vec{E}$ field is small compared to the x- and y-components. In such a case, the wave equation for the E_x component can be written as

$$-\nabla^2 E_x = k_0^2 \epsilon_r(z) E_x. \tag{8}$$

We may now express the field in a loaded resonator in terms of only TEM_{00q} modes of an empty resonator derived under the paraxial solution approximation. Let us denote by primes the quantities found using analytical formulas for the unloaded resonator. The electric field E_x in the loaded structure is approximated by

$$E_x = \sum_{n=1}^{N} a_n e'_n \tag{9}$$

where a_n are the unknown expansion coefficients and

$$e'_n = \frac{W_0}{W(z)} \exp\left(-\frac{\rho^2}{W^2(z)}\right) \sin\left[k'_{0n} z + \frac{k'_{0i}\rho^2}{2R(z)} - \zeta(z)\right] \tag{10}$$

with k'_{0i} denoting the wavenumber at the ith resonance. The expansion functions are not only exact solutions of the paraxial wave equation, but they are also approximate eigenfunctions of the ∇^2 operator. This implies that each expansion function satisfies (even though approximately) the Helmholtz equation

$$-\nabla^2 e'_n = k'^2_{0n} e'_n. \tag{11}$$

Likewise, we may assume that the approximate eigenfunctions are orthogonal, i.e.,

$$\langle e'_n, e'_k \rangle = \begin{cases} N_n, & \text{if } k = n \\ 0, & \text{if } k \neq n \end{cases} \tag{12}$$

with N_n being normalization constants. (Although, in fact, the inner product of two different approximate eigenfunctions is not exactly zero.) Substitution of (9) into (8) and then (11) into the resulting equation yields

$$\sum_{n=1}^{N} a_n k'^2_{0n} e'_n = k_0^2 \epsilon_r(z) \sum_{n=1}^{N} a_n e'_n. \tag{13}$$

Taking the inner product of the above equation with each expansion function gives the following set of equations:

$$\left\langle \sum_{n=1}^{N} a_n k'^2_{0n} e'_n, e'_k \right\rangle = k_0^2 \left\langle \epsilon_r(z) \sum_{n=1}^{N} a_n e'_n, e'_k \right\rangle. \tag{14}$$

This can be expressed in a matrix form as a generalized eigenvalue problem

$$\underline{\underline{L}}\underline{a} = k_0^2 \underline{\underline{R}}\underline{a}. \tag{15}$$

Assuming (11) and (12) hold, the left-hand-side matrix becomes simply

$$\underline{\underline{L}} = \begin{bmatrix} N_1 k_{01}'^2 & 0 & \dots & 0 \\ 0 & N_2 k_{02}'^2 & \dots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & \dots & 0 & N_N k_{0N}'^2 \end{bmatrix}.$$

This diagonal matrix can be inverted analytically and, hence, the approximate spectrum of the loaded resonator is found by solving the following standard matrix eigenvalue problem:

$$\underline{\underline{L}}^{-1}\underline{\underline{R}}\underline{a} = \delta^2 \underline{a}, \tag{16}$$

with the eigenvalues $\delta_i^2 = 1/k_{0i}^2$ related to resonant frequencies f_i by

$$f_i = c\sqrt{\delta_i^{-1}} \Big/ (2\pi). \tag{17}$$

It has to be noted that the derivation presented above included a number of assumptions, such as the lack of coupling between field components and treating each paraxial mode as an eigenfunction of the wave equation. These assumptions, while limit the accuracy of the method-of-moments solution, readily yield an approximate resonant frequency and the corresponding modal field distribution, giving an excellent starting point to a much more accurate full-wave analysis described in Section IV.

IV. Full-Wave Finite-Difference (FD) Solution

A. Discretization and Normalization of Fields

In order to describe the field behavior in the resonator in the form that is amenable to numerical treatment, we consider the FD approximation to Maxwell's equations in cylindrical coordinates. Since the resonator is assumed to have cylindrical symmetry, the angular dependence of each field component is known and, for the mode with angular number m, is expressed by either $\cos(m\phi)$ or $\sin(m\phi)$. This implies that, in order to find the quasi-TEM_{00q} modes of the hemispherical resonator, it is enough to consider Maxwell's equations for $m = 1$ in the $\rho - z$ space. To derive the eigenvalue problem, the $\rho - z$ plane is covered with Yee's mesh and the derivatives with respect to ρ and z are replaced by the modified FD formulas (see Section IV-B for the details). Apart from this modification, introduced to mitigate the effect of numerical dispersion, there are two other points pertaining to discretization that we want to underline. First, following [12], we discretize ρH_ϕ and ρE_ϕ rather than H_ϕ and E_ϕ components. Second, once the Maxwell equations have been discretized, the electric and magnetic fields are normalized by multiplying each component of the electric field by $\sqrt{\epsilon_i}$ and each component of the magnetic field by $\sqrt{\mu_i}$, where ϵ_i (μ_i) are the discretized values of the permittivity (permeability) associated with the given grid points and field components [13]. The purpose of these two operations is to obtain a discrete wave equation in the form of a standard symmetric matrix eigenvalue problem.

It has to be noted that the applied grid does not have to conform to either boundaries between dielectrics or to the curved upper mirror. Yee's mesh allows one to account for the different dielectric constants within a cell by computing the equivalent permittivity [8] and the curved metals can be handled as in [7] or [14].

Eliminating the magnetic field from the discretized Maxwell's equation yields the following matrix equivalent of the curl–curl wave equation:

$$\underline{\underline{Z}}\tilde{\underline{E}} = \omega^2 \tilde{\underline{E}} \tag{18}$$

where the tilde indicates that the electric field is normalized. The coefficient matrix $\underline{\underline{Z}}$ is highly structured and sparse with only 11 nonzero elements per row. Its spectrum consists of the eigenvalues that correspond to the resonances, as well as a large number of spurious modes that do not satisfy the divergence equation. These spurious modes are clustered around zero frequency and are a well-known problem of the curl–curl formulation in the FD technique involving all three electric- or magnetic-field components. A standard technique to prevent the numerical solver from converging to a spurious solution is by adding a scaled discrete grad–div operator to the wave equation [13], which moves the spurious eigenvalues far away from the origin. We take care of the spurious solutions by using a suitably chosen iterative matrix eigenproblem solution method, which allows us to converge on a single mode by using an appropriate starting point.

B. Accounting for Numerical Dispersion

The FD approach used in our analysis implies that the solution will be affected by numerical dispersion. Accounting for its effects is especially important while considering electrically large resonators where the dispersion is one of the main sources of error.

In order to reduce the error due to dispersion, one may refine the FD mesh at a cost of increasing the size of the eigenproblem to be solved, A different approach, which does not increase the size of the numerical problem, is based on modifying the FD scheme on Yee's mesh. The idea [15] is based on replacing the standard FD scheme operator

$$\mathbf{D}F(\eta) = \frac{F(\eta + \Delta\eta/2) - F(\eta - \Delta\eta/2)}{\Delta\eta} \tag{19}$$

with a modified operator

$$\tilde{\mathbf{D}}F(\eta) = A\frac{F(\eta + \Delta\eta/2) - F(\eta - \Delta\eta/2)}{\Delta\eta} \tag{20}$$

where A ($\neq 1$) is a certain constant, F is any of the components of the electric or magnetic field, and η is some spatial direction. If we apply the modified FD scheme (20) and Yee's mesh in cylindrical coordinates to discretize Maxwell's curl equations and then substitute the waveforms of a homogeneous TE cylindrical wave into these discretized equations, we will obtain the dispersion relation in (21), shown at the bottom of

this page, where Δz and $\Delta\rho$ define grid parameters, I denotes the index for the discretization along the ρ coordinate, B_n is an nth-order Bessel function, and k_ρ and k_z are wavenumbers in the ρ- and z-directions, respectively. In the limit ($\Delta\rho$, $\Delta z \to 0$, $I\Delta\rho \to \rho$), the above relation reduces to Bessel's equation satisfied if and only if $k_\rho^2 + k_z^2 = \omega^2/c^2$. For a given mode, provided its resonant frequency ω is known, we may compute k_ρ and then use dispersion relation (21) to obtain the value of factor A such that the value of k_z obtained from (21) will match the theoretical value of the wavenumber $k_z = \sqrt{(\omega/c)^2 - k_\rho^2}$. The modified FD scheme will then eliminate the effects of numerical dispersion for a given frequency ω.

The eigenfrequency ω of the mode (which we are looking for) is unknown. To compute factor A, we may use an approximate ω, found by some less accurate scheme, e.g., with the approximate model described in Section III, or with a full-wave solver and a standard FD scheme (19) on a coarse mesh. By using the modified FD scheme (20), we will then greatly reduce the errors associated with numerical dispersion.

In the case of a multilayered media, a different value of A can be found and applied for each medium. Also, if different FD grids are used in different media, we may apply (21) to find the corresponding values of A's.

C. Algorithm for Solving the Eigenvalue Problem

The matrix generated by the finite-difference frequency-domain (FDFD) technique is very large and the number of eigenvalues matches the size of the matrix. Since we used the curl–curl formulation, a large number of eigenvalues (clustered around zero) corresponds to spurious solutions. Moreover, the resonant frequency of interest is that of a relatively high-order mode, which implies that we have to selectively compute an eigenvalue that is located far from either end of the matrix' spectrum. Computing such interior eigenvalues is one of the most challenging task in the field of numerical linear algebra, especially when the matrix is too large to be efficiently factorized. A method that is capable of selective computation of eigenvalues also eliminates the problem of spurious eigenvalues that are present in the FDFD techniques based on the curl–curl formulation. In the past, we have proposed to use the Arnoldi method with polynomial preconditioning [16]. The preconditioner was designed as a finite-impulse response (FIR) filter whose center frequency was the approximate target eigenvalue. With the FIR filter of sufficient length, we were able to compute a few tens of modes and select the correct one by inspecting the eigenvectors. Even with this approach, reasonable solution times required parallelization of the whole algorithm and special techniques had to be devised to obtain scalability [17]. In this paper, we propose a much more efficient approach, which not only obviates the need for high-performance computing techniques, but also allows us to converge on a single mode of interest. The algorithm we propose is an adapted version of an ISIL method [10]. In a regular shift-and-invert Lanczos strategy, a few eigenvalues λ of an original eigenproblem located near a specified target σ can be computed by an iterative Lanczos algorithm applied to find the largest eigenvalues of the transformed problem

$$\left(\underline{\underline{A}} - \sigma\underline{\underline{I}}\right)^{-1}\underline{a} = \theta\underline{a} \tag{22}$$

where $\underline{\underline{I}}$ is the identity matrix. This approach involves solving a large system of equations to a good accuracy at each iteration step. High accuracy is required to prevent the breakdown of the Lanczos process. Due to the inherent ill conditioning of the matrix $\underline{\underline{A}} - \sigma\underline{\underline{I}}$, the shift-and-invert strategy becomes impractical when the matrix size exceeds a few tens of thousands. The ISIL algorithm solves the system approximately by means of an iterative technique. In this method, the accuracy of the solution step can be quite low, but the original Lanczos three-term recurrence formula has to be modified by adding an explicit projection of the eigenproblem on the Ritz vector space [10]. Below is the version of the ISIL algorithm used in our computations.

ISIL $(\underline{\underline{A}}, \underline{q}, \sigma)$
Input: $\underline{\underline{A}}$, starting vector $\underline{q}$, target σ.
Output: eigenvalue λ closest to target σ, eigenvector $\underline{x}$.
Initialization: $\underline{q} = \underline{q}/\|\underline{q}\|$, $\underline{a} = \underline{\underline{A}}\underline{q}$, $t = \underline{q}^T\underline{a}$, $\lambda = t$, $\underline{r} = \underline{a} - \lambda\underline{q}$, $\underline{\underline{Z}} = [\underline{q}]$, $\underline{\underline{Z}}_A = [\underline{a}]$, $\underline{\underline{S}} = [t]$, $\beta_o = 0$, $\underline{q}_o = \underline{q}$, $\underline{v} = \underline{q}$;
for $j = 1$ **to** m
  Solve approximately: $(\underline{\underline{A}} - \sigma\underline{\underline{I}})\underline{w} = \underline{v}$;
  Set: $\underline{w} = \underline{w} - \beta_o\underline{q}_o$, $\alpha = \underline{w}^T\underline{v}$, $\underline{w} = \underline{w} - \alpha\underline{q}$, $\underline{q}_o = \underline{q}$;
  for $i = 1$ **to** $(j-1)$
    $\underline{w} = \underline{w} - (\underline{w}^T\underline{\underline{Z}}(:,i))\underline{\underline{Z}}(:,i)$;

$$\begin{vmatrix} -\omega\mu & 0 & 0 & 0 & \dfrac{2A\sin(k_z\Delta z/2)}{\Delta z} \\ 0 & \omega\mu & 0 & \dfrac{2A\sin(k_z\Delta z/2)}{\Delta z} & 0 \\ 0 & 0 & \omega\mu B_n(\alpha) & \dfrac{-nB_n(\alpha)}{((I+1/2)\Delta\rho)^2} & \dfrac{A\left[B_n'(\gamma)(I+1) - B_n'(\delta)I\right]}{(I+1/2)\Delta r} \\ 0 & -\dfrac{2A\sin(k_z\Delta z/2)}{\Delta z} & n & -\omega\epsilon & 0 \\ -\dfrac{2AB_n'(k_\rho I\Delta\rho)\sin(k_z\Delta z/2)}{\Delta z} & 0 & -A\dfrac{B_n(\alpha) - B_n(\beta)}{\Delta\rho} & 0 & \omega\epsilon B_n'(k_\rho I\Delta\rho) \end{vmatrix} = 0,$$

where $\alpha = k_\rho(I+1/2)\Delta\rho$; $\beta = k_\rho(I-1/2)\Delta\rho$; $\gamma = k_\rho(I+1)\Delta\rho$; $\delta = k_\rho I\Delta\rho$ (21)

Set: $\beta = \|\underline{w}\|; \underline{q} = \underline{w}/\beta;\ \beta_o = \beta;$
Compute: $\underline{a} = \underline{\underline{A}}\underline{q};\ \underline{t} = \underline{q}^T\underline{\underline{Z}}_A;\ \gamma = \underline{q}^T\underline{a};$
$\underline{\underline{S}} = [\underline{\underline{S}}, \underline{t}^T; \underline{t}, \gamma];\ \underline{\underline{Z}} = [\underline{\underline{Z}}, \underline{q}],\ \underline{\underline{Z}}_A = [\underline{\underline{Z}}_A, \underline{a}];$
Make eigendecomposition $\underline{\underline{S}}\underline{\underline{U}} = \underline{\underline{U}}\underline{\underline{\Lambda}}$ of the symmetric
matrix $\underline{\underline{S}}$, select eigenvalue λ_k closest to target σ and set:
$\lambda = \lambda_k,\ \underline{x} = \underline{\underline{Z}}\underline{\underline{U}}(:, k),\ \underline{r} = (\underline{\underline{Z}}_A - \lambda\underline{\underline{Z}})\underline{\underline{U}}(:, k);$
If converged then exit with λ and $\underline{x}$.

V. Computational Results

The technique described above has been applied to calculate the resonant frequency of the quasi-$\mathrm{TEM}_{0,0,52}$ mode in the open hemispherical resonator containing a thin sapphire disc of thickness $t = 0.330$ mm and relative dielectric constant of $\epsilon_r = 9.4$ suspended at the height t_d above the planar mirror (cf. Fig. 1). The radius of curvature of the upper mirror equaled $R = 76.200$ mm and the height of the structure was $L = 71.885$ mm.[1] In order to limit the computational domain, the open resonator was modeled as a *closed* structure with sidewalls located sufficiently far from the center of the resonator, where the field intensity is negligibly small for the considered class of modes. Since the sapphire disc is very thin compared to the resonator height ($L/t \approx 218$), the structure was covered with a mesh, which was uniform in the ρ-direction and nonuniform in the z-direction. In the computations, it was assumed that the region close to the planar mirror is covered with a $10\times$ denser mesh than the remaining part of the structure. This implies that the dispersion properties for each mesh region are different. In order to use the dispersion correction algorithm, in each mesh region we used a different factor A [cf. (20)]. To eliminate the reflection caused by mesh density mismatch, we placed the boundary between the meshes at the plane where the magnitude of the E_z component is minimal. Since the location of this plane is not known *a priori*, we used the approximate field distribution obtained from the Galerkin method to determine the z coordinate for interfacing the meshes. The resonant frequencies of the quasi-$\mathrm{TEM}_{0,0,52}$ mode were computed for different distances between the disc and bottom planar mirror. They were then compared to the values of resonant frequencies measured at the Texas Center for Superconductivity, University of Houston, Houston, TX. The results are shown in Fig. 2.

One may note that computed results exceed the measurements by less than 0.1%. The results shown in this figure were computed with the grid $\Delta z = 0.16$ mm $= \lambda/17$. (In the region close to the planar mirror, we applied a finer grid $\Delta z_1 = 0.1 \cdot \Delta z$).) For these grids, the problem size equaled $N = 474\,390$. The computations were carried out in MATLAB on a 1-GHz Pentium III computer. The ISIL algorithm with the starting vector calculated via the Galerkin method required only approximately 25 min to converge. It proved to be much more efficient than the Arnoldi method with polynomial preconditioning that we used in the past [16]. With the latter algorithm, we had to resort to parallel computing to obtain results in an acceptable time [17].

[1]The length of the resonator was computed using analytical formulas for the homogeneous (empty) resonator and values of measured resonant frequencies for a few modes in that resonator.

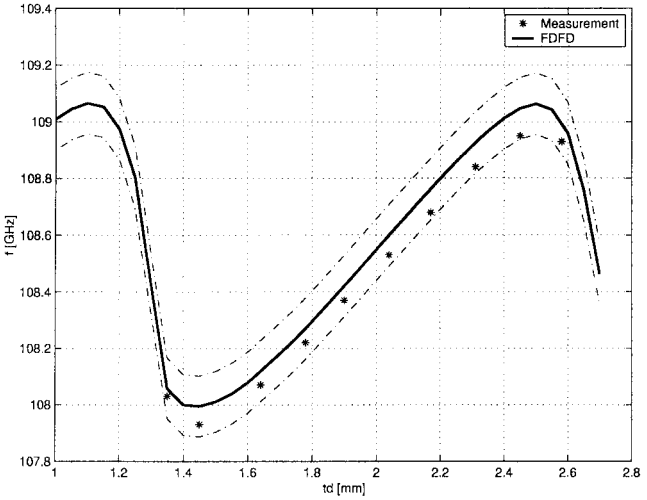

Fig. 2. Resonant frequency of the quasi-$\mathrm{TEM}_{0,0,52}$ of a hemispherical resonator loaded with a thin sapphire disc versus the distance between the disc and bottom planar mirror. Computed results: solid line, measurements: asterisks. The dashed–dotted lines define the $\pm 0.1\%$ relative error threshold with respect to the computed results.

TABLE I
Convergence of the Computed Resonant Frequency f_0 ($t_d = 2$ mm) and the Values of the Dispersion Correction Factors A_1 and A_2 Used on a Coarser Grid (Δz) and the Finer Grid ($\Delta z/10$)

Δz [mm]	$\frac{\lambda}{\Delta z}$	$\frac{\mathrm{N}}{1000}$	f_0 [GHz]	A_1	A_2
0.28	10	180	106.8554	1	1
0.28	10	180	108.6093	1.017034405	1.001571073
0.24	12	241	108.5742	1.012478721	1.001151028
0.20	14	325	108.5626	1.008642755	1.000794355
0.18	15	388	108.5568	1.006992607	1.000640339
0.16	17	474	108.5484	1.005519361	1.000503093
0.15	18	675	108.5477	1.004861916	1.000442017
0.14	20	752	108.5459	1.004233422	1.000383286
0.12	23	874	108.5431	1.003096832	1.000276081
0	∞		108.5357		

To show the influence of gridding and the dispersion correction factors, we ran the convergence tests for $t_d = 2$ mm. The results are given in Table I. With modest computer resources, we were able to easily solve problems with 874 350 unknowns. It is seen that the calculations converge uniformly from above and, as the grid is refined, the dispersion correction factors tend to one. Based on the results calculated for several grids, we determined that the order of convergence of the presented algorithm with a dispersion correction equals approximately 2.6. This means that the convergence is faster than in the standard FDFD method since, if we switch off the dispersion correction, the order of the method becomes 2.0, as predicted by the general theory and observed in practice. We used the computed convergence order in the Richardson extrapolation algorithm to determine the limit as $\lambda/\Delta z \to \infty$. This extrapolated value is shown in the last row in Table I. The first row of this table shows the results calculated for $A_1 = A_2 = 1$, i.e., without a dispersion correction. It can easily be verified that, for the coarsest grid $\lambda/\Delta z = 10$, the dispersion correction reduces the error relative to the extrapolated result from -1.549% to 0.068%, i.e., by a factor of 23.

VI. Conclusions

In this paper, we have shown that rotationally symmetric inhomogeneously loaded open hemispherical resonators could be accurately and efficiently modeled using the FDFD method. A very high accuracy of computations was achieved by coupling a conventional FDFD approach with sophisticated numerical techniques, including conformal analysis of boundaries or the algorithm for correcting the error due to numerical dispersion. Resonant frequencies of the desired high-order modes were found by solving a very large-scale symmetric eigenproblem. In order to reduce computational cost associated with solving this problem, we proposed a strategy of *selective* computation of a single desired mode based on the: 1) approximate solution with the Galerkin method and 2) application of the ISIL iterative method. Excellent agreement of the computed results with experimental data and short solution time were obtained.

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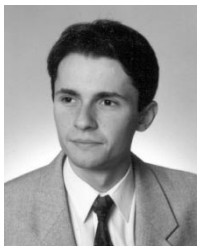

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